Math 123 – Calculus III
Practice Problems – Final Exam
Spring 2004

These problems are intended to be used as part of a review for the final exam. Do not confuse this, however, with a sample final. The problems below are not a substitute for studying all the material covered in class and the homework assignments. Review also the material for the first and second test and the midterm.

1. Consider the function $f(x, y) = \cos(x)e^{-y^2} + \cos(y)e^{-x^2}$.

   a) Find the rate of change of f at $(1, 2)$ in the direction of $\vec{v} = 3\vec{i} + 4\vec{j}$.

   b) In what direction is the rate of change of f at P greatest?

2. Find the maximum of $f(x, y) = xy$ restricted to the curve $(x+1)^2 + y^2 = 1$. Give both the coordinates of the point and the value of f.

3. Consider the sphere $x^2 + y^2 + (z-3)^2 = 25$ and the cylinder $x^2 + y^2 = 4$. Set up (but do not evaluate) an integral which calculates the volume of the solid bounded above by the sphere and on the insides by the cylinder.

4. Let $\mathbf{F}(x, y, z) = (2y, 3x, xy)$. Compute $\oint_C \mathbf{F} \cdot d\mathbf{x}$, where C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z + y = 2$ oriented counterclockwise when viewed from above. (Hint: Use Stokes' theorem.)

5. Evaluate the line integral $\int_C (\sin x \cos y + 2x)\, dx + (\cos x \sin y - 2x)\, dy$, where C is the boundary of the circle with center $(0, 0)$ and radius $\frac{1}{2}$ oriented counterclockwise.

6. Let $a, b, c \in \mathbf{R}$ such that $a, b, c \neq 0$. Which of the following are non-singular?

   (A) $\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} a & 0 & b \\ a & b & c \\ a & 0 & b \end{bmatrix}$ (C) $\begin{bmatrix} a & b & b \\ 0 & b & a \\ 0 & 0 & c \end{bmatrix}$ (D) $\begin{bmatrix} a & b & c \\ 0 & 0 & b \\ 0 & 0 & c \end{bmatrix}$

7. Set up an integral for the work done by the force $\mathbf{F} = (xz, y^2, xy)$ in moving a particle along the curve $x = e^y$,$z = 1$ from (1,0,1) to (e,1,1).

8. Let

   $$A = \begin{bmatrix} a & a^2 & 0 \\ 1 & 4 & b \\ b^2 & 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 8 & 2b \\ a & a^2 & 0 \\ b^2 - a & 3 - a^2 & 2 \end{bmatrix}.$$

   If $\det(A) = -2$, what is $\det(B^{-1})$?

9. Let $\mathbf{F}(x, y, z) = (z, y, x^2)$ and let M be the surface defined by

   $$M = \{(x, y, z) | x^2 + y^2 \leq 1, z = 0\}.$$

   Calculate $\iint_M \mathbf{F} \cdot \mathbf{n}\, d\sigma$, using the normal with the same direction that $(0, 0, 1)$.

10. Let $\mathbf{F}(x, y, z) = (x^2 - xy, xz, 4z + yz)$. Find the points for which div $\mathbf{F} = 0$.

11. Let $T : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be the linear transformation that rotates a vector 60 degrees counterclockwise. Find the matrix representation of T.

12. Let $\mathbf{F}(x, y, z) = (x, y, z)$ and let C be the unit circle in the xy-plane counterclockwise oriented. Which of the following are true?

    I. $\int_C \mathbf{F} \cdot d\mathbf{x} = 2\pi$ II. $\mathbf{F} = \vec{\nabla} f$ for some scalar function f III. $\vec{\nabla} \cdot \mathbf{F} = 0$

    (A) I only (B) II only (C) I and II (D) I, II, and III (E) III only (F) II and III only

13. Let $f, g : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$, where $g(1, 0) = (1, 1)$, $Jf(1, 1) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, and $J(f \circ g)(1, 0) = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. Find $Jg(1, 0)$.

14. Consider the vector field $\mathbf{F}(x, y, z) = (4xy + z^2, 2x^2 + 6yz, 2xz + 3y^2)$.

    a) Show that $\mathbf{F}$ is a path independent vector field.

    b) Find a function f such that $\vec{\nabla} f = \mathbf{F}$.

    c) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$, where C is any smooth curve from $(1, 2, 0)$ to $(3, 0, 1)$.

15. Set up an integral to integrate $f(x, y, z) = xz$ over the part of the surface $z = x^2 + y^2$ enclosed by $x^2 + y^2 = 4$.

16. Suppose you have a perfectly spherical peeled orange of radius 2 inches, and you give one half of the orange to your hungry Math 123 instructor. Set up a triple integral in spherical coordinates to compute the volume of the remaining portion of the orange.

17. Let $\mathbf{F}$ be the vector field on $\mathbf{R}^3$ defined as

$$\mathbf{F}(x, y, z) = 0\vec{i} + yz\vec{j} + z^2\vec{k}$$

    a) Compute $\iint_M \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where M is the upper portion of the cylinder $y^2 + z^2 = 1$ cut by the planes $z = 0$, $x = 0$, and $x = 2$. Use the outward normal.

    b) Let M be the surface of the same half cylinder with the bottom and 2 sides included. Set up an integral to compute $\iint_M \mathbf{F} \cdot \mathbf{n}\, d\sigma$. Use the outward normal.

18. Given the following matrix A and its reduced row echelon form:

$$A = \begin{bmatrix} -3 & 0 & -15 & 2 & -5 \\ -4 & -1 & -23 & 2 & -8 \\ -2 & 0 & -10 & 1 & -4 \\ 3 & 1 & 18 & 0 & 9 \end{bmatrix} \longrightarrow \text{rref} A = \begin{bmatrix} 1 & 0 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

    a) What is the rank A?

    b) What is the nullity A?

c) Find all solutions to the linear system:

$$\begin{bmatrix} -3 & 0 & -15 \\ -4 & -1 & -23 \\ -2 & 0 & -10 \\ 3 & 1 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$$

19. Compute the flux of the vector field $\mathbf{F}(x, y, z) = (2x, 2y, 2z)$ across all the sides of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. (Hint: Use the divergence theorem.)

20. The minimum and maximum of $f(x, y) = x^2 + y$ on the curve $x^2 + y^2 = 1$ occur at the points (x, y) that are solutions of the system of equations:

$$(a)\begin{cases} 2x &= 2x\lambda \\ 1 &= 2y\lambda \\ x^2 + y &= 0 \end{cases} \qquad (b)\begin{cases} 2x &= 2x\lambda \\ 1 &= 2y\lambda \\ x^2 + y^2 - 1 &= 0 \end{cases}$$

$$(c)\begin{cases} 2x &= 2x\lambda \\ 2y &= \lambda \\ x^2 + y - 1 &= 0 \end{cases} \qquad (d)\begin{cases} 2x &= 2x\lambda \\ 2y &= \lambda \\ x^2 + y^2 &= 0 \end{cases}$$

(e) None of the above.

21. Let $f(x, y, z) = x + 2y + z$, and let

$$S = \{(x, y, z) : x^2 + y^2 + z^2 \le 4,\ \sqrt{x^2 + y^2} \le z\}.$$

(S is the solid bounded on the sides by the cone $z = \sqrt{x^2 + y^2}$ and on the top by the sphere $x^2 + y^2 + z^2 = 4$.) **SET UP** an integral to compute

$$\iiint_S f(x, y, z)\, dV,$$

a) using rectangular coordinates (x, y, z). DO NOT COMPUTE THE INTEGRAL.

b) using cylindrical coordinates (θ, r, w). DO NOT COMPUTE THE INTEGRAL.

22. For each of the following vector fields, determine if the vector field is conservative or not. MAKE SURE YOU JUSTIFY YOUR ANSWER AS FOLLOWS: If the vector field is conservative, find a potential function. If the vector field is not conservative, explain why you can conclude that.

a) $\mathbf{F}(x, y, z) = (2xy^2, x^2y, xyz)$.

b) $\mathbf{G}(x, y) = (2xy + 2, x^2 + 1)$.

23. Consider the cardioid curve given by $r = 1 + \cos(t)$, $0 \le t \le 2\pi$. USING GREEN'S THEOREM, find the line integral of the vector field $\mathbf{F} = (\sin(y)e^x, 2x + e^x\cos(y))$ counterclockwise around the cardioid.

24. Let $\mathbf{u}(x, y) = (x - y, y^3 + y, x)$ and $\mathbf{f}(s, t) = (s^2 + t^2, st)$.

   a) Compute the Jacobian matrix of $\mathbf{u}$ at (x, y).

   b) Compute the Jacobian matrix of $\mathbf{f}$ at (s, t).

   c) Let $\mathbf{v}(s, t) = (v_1(s, t), v_2(s, t), v_3(s, t)) = (\mathbf{u} \circ \mathbf{f})(s, t)$. Compute $\dfrac{\partial v_2}{\partial t}(s, t)$ in terms of s and t.

25. Let $\mathbf{T} : \mathbf{R}^3 \to \mathbf{R}^2$ and $\mathbf{S} : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformations

$$\mathbf{T}(x_1, x_2, x_3) = (-2x_2, -2x_1) \text{ and } \mathbf{S}(x_1, x_2) = (x_1, x_2, 0).$$

   a) Find a matrix A that represents the linear transformation $\mathbf{T}$.

   b) Find a matrix B that represents the linear transformation $\mathbf{S}$.

   c) Find a matrix C that represents the linear transformation $\mathbf{R} = \mathbf{T} \circ \mathbf{S}$.

   d) Find the area of the image of the square $\{(x_1, x_2) \,|\, 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq 1\}$ under the linear transformation $\mathbf{R}$ of part C).

26. Let $\mathbf{F}(x, y, z) = (y, x, 1)$.

   a) Find a potential for $\mathbf{F}(x, y, z)$.

   b) Find $\int_C \mathbf{F} \cdot d\mathbf{x}$, where C is given by the union of the segments from $(0, 0, 0)$ to $(1, 1, 2)$, from $(1, 1, 2)$ to $(1, 3, 2)$, from $(1, 3, 2)$ to $(2, 3, 2)$, from $(2, 3, 2)$ to $(2, 3, 5)$. (Hint: Use the fundamental theorem of line integrals.)

27. Compute the mass of surface given by

$$\mathbf{f}(u, v) = (u, u^3 + v, v)$$

   for $0 \leq u \leq 2$ and $0 \leq v \leq 3$, if the density of mass (in units of mass per unit of area) is given by the function $g(x, y, z) = x$.

28. From the book:

   Sec. B6.3, p404: 1, 2, 3, 4, 5, 14, 15.

   Sec. B6.4, p414: 2, 3, 6, 7, 8.