Homework 1

Math 766
Spring 2012

7.1.3 Suppose that for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ is bounded. If $f_n \to f$ uniformly on E as $n \to \infty$, then $\{f_n\}$ is uniformly bounded on E and f is a bounded function on E.
Proof: For each $n \in \mathbb{N}$, there exists $M_n > 0$ such that $|f_n(x)| \le M_n$ for all $x \in E$. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that

$$n, m \ge N \Longrightarrow |f_n(x) - f_m(x)| < 1 \text{ for all } x \in E.$$

Define $M = \max(M_1, M_2, \dots, M_N) + 1$. Then for any $n \in \mathbb{N}$ and any $x \in E$, it follows that $|f_n(x)| \le M$ since

$$\begin{aligned} n > N &\Longrightarrow |f_n(x)| \le |f_N(x)| + |f_n(x) - f_N(x)| \le M_N + 1 \le M \\ n \le N &\Longrightarrow |f_n(x)| \le M_n \le M. \end{aligned}$$

Therefore f_n is uniformly bounded. Since $f_n \to f$ uniformly on E, there exists $N \in \mathbb{N}$ such that

$$n \ge N \Longrightarrow |f_n(x) - f(x)| < 1 \text{ for all } x \in E.$$

Then for all $x \in E$

$$|f(x)| \le |f_N(x)| + |f(x) - f_N(x)| \le M_N + 1.$$

Therefore f is bounded on E as well. □

7.1.5 Suppose $f_n \to f$ and $g_n \to g$ uniformly on $E \subset \mathbb{R}$ as $n \to \infty$.

c) If f and g are bounded on E, then $f_n g_n \to fg$ uniformly on E.
Proof: Let $\varepsilon > 0$ since f, g are bounded on E, let $M > 0$ such that $|f(x)| \le M$ and $|g(x)| \le M$ for all $x \in E$. There exists $N \in \mathbb{N}$ such that

$$n \ge N \Longrightarrow |f_n(x) - f(x)| < \varepsilon \text{ for all } x \in E$$

$$n \ge N \Longrightarrow |g_n(x) - g(x)| < \varepsilon \text{ for all } x \in E.$$

This implies also that for $n \ge N$ and $x \in E$

$$|f_n(x)| \le |f(x)| + |f(x) - f_n(x)| \le M + \varepsilon.$$

Then for all $x \in E$ and for any $n \ge N$

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\le |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)|\,|g_n(x) - g(x)| + |f_n(x) - f(x)|\,|g(x)| \\ &= (M + \varepsilon)\varepsilon + \varepsilon M \\ &= (2M + \varepsilon)\varepsilon. \end{aligned}$$

Therefore $f_n g_n \to fg$ uniformly on E as $n \to \infty$. □