Homework 2

Math 766
Spring 2012

7.2.3 Let $E(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

a) Prove that the series defining $E(x)$ converges uniformly on any closed interval $[a,b]$.
Proof: Let $[a,b] \subset \mathbb{R}$ be a closed interval and define $M = \max(|a|,|b|)$. Then $\frac{|x|^k}{k!} \leq \frac{M}{k!}$ and

$$\sum_{k=0}^{\infty} \frac{M}{k!}$$

converges by the ratio test since

$$\lim_{k\to\infty} \frac{M/(k+1)!}{M/k!} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$$

Then by the Weirstrass M-test, $\sum_{k=0}^{\infty}$ converges absolutely and uniformly on $[a,b]$. □

b) Prove that

$$\int_a^b E(x)dx = E(b) - E(a)$$

for all $a,b \in \mathbb{R}$. *Proof:* Let $a,b \in \mathbb{R}$. By part **a)**, the series defining E converges uniformly on $[\min(a,b),\max(a,b)]$. Also $x^k/k!$ is continuous on $[\min(a,b),\max(a,b)]$ and hence integrable on $[\min(a,b),\max(a,b)]$. Then integrating term by term is permissible, so by the fundamental theorem of calculus

$$\begin{aligned}
\int_a^b E(x)dx &= \sum_{k=0}^{\infty} \int_a^b \frac{x^k}{k!}dx \\
&= \sum_{k=0}^{\infty} \left(\frac{b^{k+1}}{(k+1)!} - \frac{a^{k+1}}{(k+1)!} \right) \\
&= \sum_{k=0}^{\infty} \frac{b^{k+1}}{(k+1)!} - \sum_{k=0}^{\infty} \frac{a^{k+1}}{(k+1)!} \qquad \text{(since both sums converge)} \\
&= 1 + \sum_{k=1}^{\infty} \frac{b^k}{k!} - \left(1 + \sum_{k=1}^{\infty} \frac{a^{k+1}}{k!} \right) \\
&= \sum_{k=0}^{\infty} \frac{b^k}{k!} - \sum_{k=0}^{\infty} \frac{a^k}{k!} \\
&= E(b) - E(a).
\end{aligned}$$

Note that we can split the limits in the third equality since both of these limits exist. □

(c) Prove that the function $y = E(x)$ satisfies the initial value problem

$$y' - y = 0, \qquad y(0) = 1.$$

Proof: Fix $x \in \mathbb{R}$ and take $R > |x|$. Consider

$$\sum_{k=0}^{\infty} \frac{d}{dx}\frac{x^k}{k!} = 0 + \sum_{k=1}^{\infty} k\frac{x^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This series converges uniformly on $(-R,R)$ by part **(a)**. Also by part **(a)** the series defining $E(0)$ converges

$$E(0) = \sum_{k=0}^{\infty} \frac{0^k}{k!} = 1.$$

Therefore one may differentiate term by term to compute

$$y'(x) = \frac{d}{dx}\sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dx}\frac{x^k}{k!} = \sum_{k=0}^{\infty} k\frac{x^{k-1}}{(k-1)!} \sum_{k=0}^{\infty} \frac{x^k}{k!} = E(x) = y.$$

Then $y = E(x)$ satisfies the initial value problem: $y' - y = 0$; $y(0) = 1$. $\square$