Homework 3

Math 766
Spring 2012

7.3.2 Find the interval of convergence of the following power series.

b) $\sum_{k=0}^{\infty}((-1)^k+3)^k(x-1)^k$.

Solution: First compute the radius of convergence for this power series

$$R = \frac{1}{\limsup_{k\to\infty}(-1)^k+3} = \frac{1}{4}.$$

So the power series converges on $(\frac{3}{4}, \frac{5}{4})$, and it only remains to test the endpoints, $\frac{3}{4}$ and $\frac{5}{4}$. Computing the terms of the series at the endpoint $\frac{3}{4}$,

$$((-1)^k+3)^k\left(\frac{3}{4}-1\right)^k = \frac{(-1)^{k+1}-3}{4}.$$

This sequence does not tend to 0 as $k \to \infty$, so the power series above does not converge at $x = \frac{3}{4}$. Similarly for the endpoint $x = \frac{5}{4}$, the terms of the series are

$$((-1)^k+3)^k\left(\frac{5}{4}-1\right)^k = \frac{(-1)^k+3}{4}$$

which again does not tend to 0 as $k \to \infty$. Then the power series does not converge at $x = \frac{5}{4}$ either. Therefore the interval of convergence is $(\frac{3}{4}, \frac{5}{4})$.

7.3.6 **a)** Prove that if $\sum_{k=0}^{\infty} a_k$ converges to L, then $\sum_{k=0}^{\infty} a_k$ is Able summable to L.

Proof: Since $\sum_{k=0}^{\infty} a_k$ converges, $a_k \to 0$ as $k \to \infty$ and hence a_k is bounded. Therefore the power series $f(x) := \sum_{k=0}^{\infty} a_k x^k$ converges absolutely for $x \in (-1,1)$. Since $\sum_{k=0}^{\infty} a_k = L$, the power series defining $f(x)$ converges on $(-1,1]$. Then by Abel's Theorem, f is continuous on $[0,1]$. Therefore

$$\lim_{r\to 1^-}\sum_{k=0}^{\infty} a_k r^k = \lim_{r\to 1^-} f(r) = f(1) = \sum_{k=0}^{\infty} a_k = L.$$

Then $\sum_{k=0}^{\infty} a_k$ is Abel summable to L. □

b) Find the Abel sum of $\sum_{k=0}^{\infty}(-1)^k$.
Solution: Using the convergence of geometric series, compute

$$\begin{aligned}\lim_{r\to 1^-}\sum_{k=0}^{\infty}(-1)^k r^k &= \lim_{r\to 1^-}\sum_{k=0}^{\infty}(-r)^k\\ &= \lim_{r\to 1^-}\frac{1}{1+r}\\ &= \frac{1}{2}.\end{aligned}$$