Homework 5

Math 766
Spring 2012

10.3.8 Let Y be a subspace of X.

a) Show that V is open in Y if and only if there is an open set U in X such that $V = U \cap Y$.

Note that Y is given a topology in two different ways here: One topology is the metric space topology on Y generated by open balls under the metric $\rho|_{Y\times Y}$ (described on page 350). The second topology on Y in question here is the subspace topology inherited from X, which defines the open sets of Y to be of the form $U \cap Y$ for U open in X (defined on page 368). The point of this exercise is to prove that these two topologies are in fact the same.

Proof: $(\Rightarrow)$ Let U be and open set in the metric space $(Y, \rho_{Y\times Y})$. Then for each $y \in U$, there exists $r_y > 0$ such that $B^Y_{r_y}(y) \subset U$ where $B^Y_{r_y}(y) = \{u \in Y : \rho|_{Y\times Y}(y,u) < r_y\}$. It follows that

$$V = \bigcup_{y\in V} B^Y_{r_y}(y).$$

Now define

$$U = \bigcup_{y\in V} B^X_{r_y}(y), \quad \text{where} \quad B^X_{r_y}(y) = \{x \in X : \rho(x,y) < r_y\}.$$

Then U is open in (X, ρ) and

$$U \cap Y = \left(\bigcup_{y\in V} B^X_{r_y}(y)\right) \cap Y = \bigcup_{y\in V} B^Y_{r_y}(y) = V$$

$(\Leftarrow)$ Assume that $V = U \cap Y$ for some set U that is open in (X, ρ). Then for each $x \in U$ there exists $r_x > 0$ such that $B^X_{r_x}(x) \subset U$ and U can be written

$$U = \bigcup_{x\in U} B^X_{r_x}(x).$$

Now since for each $y \in V$, $y \in U$ as well. Then $B^X_{r_y}(y) \subset U$,

$$B^Y_{r_y}(y) = B^X_{r_y}(y) \cap Y \subset U \cap Y = V,$$

and $B^Y_{r_y}(y)$ is open in the metric topology of $(Y, \rho_{Y\times Y})$. Then it follows that V is open in $(Y.\rho_{Y\times Y})$. $\square$

b) Show that E is closed in Y if and only if there is a closed set A in X such that $E = A \cap Y$.
Proof: Using part **a)**, given a subset $E \subset Y$ and the definition of a closed set,

$$\begin{aligned}
E \text{ is closed in } (Y, \rho|_{Y\times Y}) &\iff Y \backslash E \text{ is open in } (Y, \rho|_{Y\times Y}) \\
&\iff Y \backslash E = U \cap Y \text{ for some } U \text{ open in } (X, \rho) \\
&\iff Y \backslash E = (X \backslash A) \cap Y \text{ for some } A \text{ closed in } (X, \rho) \\
&\iff Y \backslash E = Y \backslash (A \cap Y) \text{ for some } A \text{ closed in } (X, \rho) \\
&\iff E = A \cap Y \text{ for some } A \text{ closed in } (X, \rho)
\end{aligned}$$

□