Homework 6

Math 766
Spring 2012

10.4.2 Let A and B be compact subsets of X. Prove that $A \cup B$ and $A \cap B$ are compact.
Proof: Let $\mathcal{U} = \{u_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $A \cup B$. Then $\mathcal{U}$ is also an open cover of A and B since

$$A \subset A \cup B \subset \bigcup_{\alpha\in\Lambda} U_\alpha$$

$$B \subset A \cup B \subset \bigcup_{\alpha\in\Lambda} U_\alpha.$$

Since A and B are compact, there exist finite subcovers $\{U_1,...,U_n\} \subset \{U_\alpha\}_{\alpha\in\Lambda}$ and $\{U_{n+1},...,U_N\} \subset \{U_\alpha\}_{\alpha\in\Lambda}$ of A and B respectively. That is

$$A \subset \bigcup_{j=1}^{n} U_j \qquad\qquad B \subset \bigcup_{j=n+1}^{N} U_j.$$

Then it follows that the finite collection $\{U_1,U_2,...,U_N\} \subset \{U_\alpha\}_{\alpha\in\Lambda}$, is in fact an open cover of $A \cup B$,

$$A \cup B \subset \bigcup_{j=1}^{n} U_j \cup \bigcup_{j=n+1}^{N} U_j = \bigcup_{j=1}^{N} U_j.$$

Therefore $A \cup B$ is compact.
Now we prove that $A \cap B$ is compact. Since A and B are compact, they are closed. Then $A \cap B$ is closed as well. But then $A \cap B \subset A$ is a closed subset of the compact set A, and hence is a compact set. □

10.4.8 **a)** If $H_1, H_2, ...$ is a nested sequence of nonempty compact sets in X, then

$$\bigcap_{k=1}^{\infty} H_k \neq \emptyset.$$

Proof: For a contradiction, assume that

$$\bigcap_{k=1}^{\infty} H_k = \emptyset.$$

Then it follows that $\{U_k = X\backslash H_k\}$ is an open cover of X, and in particular an open cover of H_1. Then there exist $\{U_1,...,U_N\}$ such that

$$H_1 \subset \bigcup_{j=1}^{N} U_j = \bigcup_{j=1}^{N} X\backslash H_j = X\backslash \bigcap_{j=1}^{N} H_j = X\backslash H_N.$$

On the other, recall that $H_1 \supset H_N$, so

$$H_N \subset H_1 \subset X\backslash H_N.$$

But this implies that $H_N = \emptyset$, which is a contradiction. Therefore

$$\bigcap_{k=1}^{\infty} H_k \neq \emptyset.$$

□

b) Prove that $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is closed and bounded but not compact in the metric space $\mathbb{Q}$ introduced in Example 10.5.
Proof: We show that $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is closed by showing that its complement in $\mathbb{Q}$, $\mathbb{Q}\backslash(\sqrt{2},\sqrt{3})$, is open. So let $x \in \mathbb{Q}\backslash(\sqrt{2},\sqrt{3})$. Then $x \notin (\sqrt{2},\sqrt{3})$ and since $x \in \mathbb{Q}$ we can even say that $x \notin [\sqrt{2},\sqrt{3}]$. Let $\varepsilon = \min(|x-\sqrt{2}|,|x-\sqrt{3}|)$ which is larger than 0. It also follows that $(x-\varepsilon,x+\varepsilon)\cap\mathbb{Q} \subset \mathbb{Q}\backslash(\sqrt{2},\sqrt{3})$ and that $\mathbb{Q}\backslash(\sqrt{2},\sqrt{3})$ is open. Then $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is closed. It is clear that $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is bounded, for example we may take $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}B(0,\sqrt{3})$. To see that $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is not compact, consider the collection $U_n = (\sqrt{2}+1/n,\sqrt{3})\cap\mathbb{Q}$. For each $x \in U_n$, $(x-\varepsilon,x+\varepsilon)\cap\mathbb{Q} \subset U_n$ where $\varepsilon = \min(|\sqrt{2}+1/n-x|,|x-\sqrt{3}|)$. So U_n is open for each $n \in \mathbb{N}$. For each $x \in (\sqrt{2},\sqrt{3})\cap\mathbb{Q}$, $x-\sqrt{2} > 0$. So there exists $N \in \mathbb{N}$ such t hat $\frac{1}{N} < x-\sqrt{2}$. Therefore $x \in (\sqrt{2}+1/N,\sqrt{3})\cap\mathbb{Q}$. Therefore

$$(\sqrt{2},\sqrt{3})\cap\mathbb{Q} \subset \bigcup_{n=1}^{\infty} U_n.$$

If $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is compact, then there exists a finite subcover

$$(\sqrt{2},\sqrt{3})\cap\mathbb{Q} \subset \bigcup_{n=1}^{N} U_n.$$

By by the density of the rational numbers, there exists $x \in (\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ such that $x-\sqrt{2} < \frac{1}{N}$. Therefore $x \notin (\sqrt{2}+1/N,\sqrt{3}) = U_N$. Note that $U_{k+1} \supset U_k$ for all $k \in \mathbb{N}$ and we have that $x \notin U_n$ for any $n \leq N$. But this contradicts that $\{U_1,...,U_N\}$ is an open cover of $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$. Therefore $(\sqrt{2},\sqrt{3})\cap\mathbb{Q}$ is not compact. □

c) Show that Cantor's Intersection Theorem does not hold in an arbitrary metric space if compact is replaced with closed and bounded.
Proof: Define

$$H_k = (\sqrt{2},\sqrt{2}+1/k)\cap\mathbb{Q}.$$

Applying the argument from part **a)**, H_k is closed and bounded for each k. Now for a contradiction, assume that

$$x \in \bigcap_{k=1}^{\infty} H_k.$$

Then for all $k \in \mathbb{N}$

$$\sqrt{2} < x < \sqrt{2} + \frac{1}{k}.$$

Then by the squeeze theorem, $x = \sqrt{2} \notin (\sqrt{2}, \sqrt{2}+1) = H_1$, which is a contradiction. Therefore

$$\bigcap_{k=1}^{\infty} H_k = \emptyset$$

and Cantor's Intersection Theorem does not hold if we replace compact with closed and bounded. □