HW 3

BRIEF SOLUTIONS TO SELECTED Pb's

a) • $f(x) \le f(y)$ FOR $x < y$

$$\Rightarrow g(x) = (f(x)+x)/2 \le (f(y)+x)/2 < (f(y)+y)/2 = g(y)$$

• g IS CONTINUOUS BECAUSE IS LINEAR COMBINATION OF CONTINUOUS FUNCTIONS

• $g(0) = 0$, $g(1) = 1$. BY INT. VALUE THM. g IS ONTO.

b) g^{-1} CONTINUOUS $\Rightarrow [g^{-1}]^{-1}(C) = g(C)$ IS CLOSED AND HENCE m.

$$g([0,1]) = g(C) \cup g([0,1]-C)$$

$$[0,1]-C = \bigcup_{n,k} G_{n,k}, \quad G_{n,k} = \left(\frac{k}{3^n}, \frac{k+1}{3^n}\right)$$

$$g(G_{n,k}) = \left(g\left(\frac{k}{3^n}\right), g\left(\frac{k+1}{3^n}\right)\right)$$

$$= \left(\frac{1}{2}\left(f(k/3^n) + k/3^n\right), 1/2\left(f\left(\frac{k+1}{3^n}\right) + \frac{k+1}{3^n}\right)\right)$$

BUT $f(k/3^n) = f\left(\frac{k+1}{3^n}\right)$

$$\Rightarrow m(g(G_{n,k})) = \frac{1}{2}\frac{1}{3^n} \quad \forall k = 1, \dots, 2^{n-1}$$

$$\Rightarrow \quad m\left(g([0,1]-C)\right) = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}}\frac{1}{3^n}$$

$$m\left(g([0,1]-C\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n} = 1/2$$

$$\Rightarrow m(g(C)) = 1/2$$

c) LET $N \subset g(C)$, N NON-MEASURABLE

NOW $g^{-1}(N) \subset C$ AND $m(C) = 0$

$\Rightarrow g^{-1}(N)$ IS MEASURABLE

TAKE $M = g^{-1}(N)$

d) IF M WERE BOREL, THEN

$[g^{-1}]^{-1}(M) = N$ WOULD BE MEASURABLE

e) IT IS EASY TO SEE THAT

$\chi_M \circ g^{-1} = \chi_N$

IN PARTICULAR, $\chi_M \circ g^{-1} > 0 = N$

WHICH IS NON-MEASURABLE.