MATH 810 – REAL ANALYSIS

HW 1. Outer measure and Lebesgue measure in $\mathbb{R}$

Due 09/16/15

1. Let $\mathcal{A}$ be an algebra of subsets of X.
a) Show that $\emptyset$ and X are in $\mathcal{A}$.
b) Show that if A and B are in $\mathcal{A}$, then $A \cap B$ is in $\mathcal{A}$.
c) Show that if $\mathcal{A}$ is a σ–algebra and $\{A_n\}_{n=1}^{\infty}$ is a sequence of set in $\mathcal{A}$, then $\cap_{n=1}^{\infty} A_n$ is also in $\mathcal{A}$.

2. a) Show that the intersection of σ–algebras is a σ–algebra.
b) Show that given a family $\mathcal{E}$ of subsets of X, there exists a unique smallest σ–algebra containing $\mathcal{E}$. (Note that $\mathcal{P}(X)$ is a σ–algebra containing $\mathcal{E}$.)

3. Let $A = [0, 1] \cap \mathbf{Q}$. Show that if $\{I_n\}$ is a finite collection of open intervals that covers A, then $\sum |I_n| \geq 1$. Is the same true for any infinite collection?

4. a) Show that that if A is a set with $\mu^*(A) = 0$, then A is measurable.
b) Show that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$ for any B.
c) Show that every non-empty open set of $\mathbf{R}$ has positive measure.
d) Show that every compact set of $\mathbf{R}$ has finite measure.

5. a) Prove that μ^* is translation invariant. That is $\mu^*(A) = \mu^*(A + x)$ for any point x.
b) Prove that m is translation invariant.

6. Prove that if A_1 and A_2 are measurable sets, then

$$m(A_1 \cap A_2) + m(A_1 \cup A_2) = m(A_1) + m(A_2).$$

Extra credit:

7. Review the construction of the set N that we did the first class and the (mutually disjoint) sets N_r obtained from it for $r \in [0, 1) \cap \mathbf{Q}$.
a) Show that

$$\mu^*(\cup_r N_r) < \sum_r \mu^*(N_r)$$

(so, in particular, the sets N_r are not measurable).
b) Show that any measurable subset of N must have measure zero.