MATH 810 – REAL ANALYSIS

HW 2. Abstract measure spaces

Due 9/30/15

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that for any $E, F \in \mathcal{M}$, $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E \in \mathcal{M}$.
a) For $A \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$. Show that μ_E is a measure on $(X, \mathcal{M})$.
b) Define $\mathcal{M}_E = \{F \in \mathcal{M} : F \subset E\}$. Show that $\mathcal{M}_E$ is a σ-algebra of subsets of E and that $(E, \mathcal{M}_E, \mu|_{\mathcal{M}_E})$ is a measure space. (Note that $\mu|_{\mathcal{M}_E} = \mu_E|_{\mathcal{M}_E}$.)

3. a) Let X be a set equipped with a σ-algebra $\mathcal{M}$. A function $\mu : \mathcal{M} \to [0, \infty]$ is called finitely additive if

i) $\mu(\emptyset) = 0$,

ii) and if $E_1, E_2, \cdots E_n$ are disjoint sets in $\mathcal{M}$, then $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$.

Let X be an infinite set and $\mathcal{M} = \mathcal{P}(X)$. Define $\mu(E) = 0$ if E is finite (or empty) and $\mu(E) = \infty$ if E is infinite. Show that μ is finitely additive but not a measure.

4. Let X be an uncountable set. Let

$$\mathcal{M} = \{E \subset X : E = \emptyset,\ E = X,\ E \text{ is countable or } E^c \text{ is countable}\}.$$

Define $\mu(E) = 0$ if E is empty or countable, and $\mu(E) = 1$ if E^c is countable. Show that $(X, \mathcal{M}, \mu)$ is a measure space.

5. Let $\mu_1, \mu_2, \cdots \mu_n$ be measures on $(X, \mathcal{M})$. Show that for any $a_1, a_2, \cdots, a_n \in [0, \infty)$, $\sum_{i=1}^n a_i\mu_i$ is a measure on $(X, \mathcal{M})$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and let $\bar{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N, N \in \mathcal{N}\}$. Show that $\bar{\mathcal{M}}$ is a σ-algebra. Show also that $\bar{\mu}$ defined on $\bar{\mathcal{M}}$ by $\bar{\mu}(E \cup F) = \mu(E)$ is the unique **complete** measure that extends μ to $\bar{\mathcal{M}}$.

Extra credit:

7. Let X be a non-empty set, $\mathcal{M} = \mathcal{P}(X)$, and $f : X \to [0, \infty]$ an arbitrary function. Define for $E \in \mathcal{M}$, $E \neq \emptyset$,

$$\mu(E) = \sum_{x \in E} f(x) \equiv \sup\{\sum_{x \in F} f(x) : F \subset E, F \text{ finite}\},$$

and $\mu(\emptyset) = 0$.

a) Show that $(X, \mathcal{M}, \mu)$ is a measure space.

b) Show that μ is σ-finite if and only if $f(x) < \infty$ for every $x \in X$ and $\{x : f(x) > 0\}$ is countable. (Hint: To show that μ σ-finite implies $\{x : f(x) > 0\}$ countable, use $\{x : f(x) > 0\} = \cup_{n=1}^\infty \{x : f(x) > 1/n\}$.)