MATH 810 – REAL ANALYSIS

HW 4. Integration

Due 10/28/15

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^+$. Define $\nu(E) = \int_E f\, d\mu$ for $E \in \mathcal{M}$. Show that ν is a measure and that $\int g\, d\nu = \int fg\, d\mu$ for all $g \in L^+$. (Hint: Prove the last statement for simple functions and use a limiting argument.) This is an important example of measure space, since many measures in applications arise in this way.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^+$ with $\int f < \infty$. Show that the set $\{x \in X : f(x) = +\infty\}$ has measure zero. (Hint: $f_{=\infty} = \cap_{n \geq 1} f_{>n}$; show that $\lim_{n\to\infty} \mu(f_{>n})$ cannot be positive.)

3. a) Let $(X, \mathcal{M}, \mu)$ be a measure space, $X = \cup E_n$, with E_n measurable and $E_n \subset E_{n+1}$. Then, for all $f \in L^+$, $\int f\, d\mu = \lim_{n\to\infty} \int_{E_n} f\, d\mu$.

b) Let $f : R \to [0, \infty)$ be defined by $f(x) = 1/x^2$ if $x \geq 1$ and $f(x) = 0$ otherwise. Show that the Lebesgue integral of f over R is finite.

4. Consider the measure space $(N, \mathcal{P}(N), \mu)$, where N is the set of positive integers and μ is the counting measure. Determine whether each of the following functions (sequences) $f : N \to R$ is integrable with respect to μ and whether the series $\sum_n f(n)$ is convergent. Justify your answer.

a) $f(n) = \cos(n\pi)/n$.
b) $f(n) = (-2)^{-n}$.

5. Prove the following well-known result about series of functions using results from integration in a measure space. Let $\{f_n\}$ be a sequence of continuous and differentiable functions on the interval $[a, b]$. Assume that for each t in $[a, b]$, $\sum_n |f_n(t)|$ is convergent and $|f'_n(t)| \leq g(n)$, where $\sum_n g(n)$ is convergent. Show that the function $F(t) = \sum_n f_n(t)$ is differentiable on (a, b) and $F'(t) = \sum_n f'_n(t)$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable function on X that converges almost everywhere to a function f. Show that if for some constant $C > 0$, $|f_n(x)| \leq C$ for all x and n, then

$$\int_X f(x)\, d\mu(x) = \lim_{n\to\infty} \int_X f_n(x)\, d\mu(x).$$

(Hint: This is a simple version of Lebesgue's dominated convergence theorem.)