MATH 810 – REAL ANALYSIS

HW 6. More on R^n and products of measures

Due 11/30/15

1. a) Show that the function

$$F(y) = \int_R \frac{1}{1+x^2} \frac{\sin(y-x)}{1+(y-x)^2} \, dm(x)$$

is continuous.

b) Is

$$f(x,y) = \frac{1}{1+x^2} \frac{\sin(y-x)}{1+(y-x)^2}$$

Lebesgue integrable in R^2? (Hint: remember that Lebesgue integration is translation invariant.)

2. Let $B^n = \{x \in R^n : |x| \le 1\}$ be the unit ball in R^n. Show that $m(B^n) = \pi^{n/2}/\Gamma(n/2+1)$, where m is the Lebesgue measure in R^n. (Recall that $\Gamma(x+1) = x\Gamma(x)$, and $\pi^{n/2} = \int_{R^n} e^{-|x|^2} dm(x)$.)

3. Show that for any $\epsilon > 0$ the function $f(x) = (1+|x|^{n+\epsilon})^{-1}$ is integrable in R^n with respect to the Lebesgue measure.

4. Show that for any $\epsilon > 0$ the function $f(x) = |x|^{-n+\epsilon}\chi_K(x)$, where K is a compact set, is integrable in R^n with respect to the Lebesgue measure.

5. Show that

$$\sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{2^{-(1+\epsilon)n}}{(1+2^{-n}|k|)^2} < \infty,$$

for any $0 < \epsilon < 1$, but that

$$\sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{2^{-n}}{(1+2^{-n}|k|)^2}$$

is divergent. Justify your computations.

Hint: Prove and use the inequalities

$$c_1 \le \int_1^\infty \frac{2^{-n}}{(1+2^{-n}x)^2} \, dx \le \sum_{k=-\infty}^{\infty} \frac{2^{-n}}{(1+2^{-n}|k|)^2} \le 2^{-n} + 2\int_0^\infty \frac{2^{-n}}{(1+2^{-n}x)^2} \, dx \le C_2,$$

where the constants c_1 and C_2 are independent of $n \ge 0$.