MATH 890 – FOURIER ANALYSIS – F13

HW 1. More on spaces of smooth functions

1. a) Recall that in the topology of $\mathcal{D}(R^n)$, a sequence $\{\phi_j\}$ converges to zero if there exist a compact set K so that supp $\phi_j \subset K$ for all j and $\partial^\alpha \phi_j$ converges uniformly to zero for each multi-index α. Show that if f is in $C^\infty(R^n)$ then the linear map $\phi \to f\phi$ is continuous from $\mathcal{D}$ to $\mathcal{D}$. (Hint: Remember that in $\mathcal{D}$ it is enough to check the continuity on sequences.)

b) Consider the same problem in $\mathcal{S}(R^n)$, but with f in $C^\infty(R^n)$ and "tempered at infinity", that is $|\partial^\alpha f(x)| \leq C_\alpha(1+|x|)^{N_\alpha}$ for all multi-indices α. Show that the linear map $\phi \to f\phi$ is continuous from $\mathcal{S}$ to $\mathcal{S}$.

2. This exercise will show that if we give the vector space $\mathcal{D}$ the topology induced by the family of semi-norms

$$\|\phi\|_m = \sup_{x\in R^n, |\alpha|\leq m} |\partial^\alpha \phi(x)|,$$

then it is not complete.

Proceed as follows. Fix $\phi \in \mathcal{D}(R)$ and define

$$f_N(x) = \sum_{j=1}^{N} j^{-2}\phi(x-j).$$

Show that $\{f_N\}$ is a Cauchy sequence in $\| \, \|_m$ for each m, but that $\{f_N\}$ does not converge to a function with compact support. (Hint: A picture may help you see what is going on.)

3. This exercise will show that $\mathcal{D}$ with the topology mentioned in problem 1 is not metrizable.

Proceed as follows. Consider a sequence of compact sets E_j such that $R^n = \cup E_j$ and $E_j \subset \operatorname{int}(E_{j+1})$. Consider also a sequence of functions $\{\phi_j\}$ in $\mathcal{D}$ such that supp $\phi_j \subset E_{j+1}$ and $\phi_j \equiv 1$ on E_j. Suppose now that $\mathcal{D}$ admits a metric d compatible with the topology mentioned in problem 1. Let $\{B_j\}$ be the basis of nbhd's of zero in that metric given by $B_j = \{\phi \in \mathcal{D} : d(\phi, 0) < j^{-1}\}$. Then, there exists a sequence of positive constants $\{c_j\}$ such that $c_j\phi_j \in B_j$ (this is because multiplication by scalars is continuous in $\mathcal{D}$). Arrive now at a contradiction by looking at the sequence $\{c_j\phi_j\}$.

4. Let $f(x) = e^{x^2}$. For $\varphi \in \mathcal{D}(R)$, define $L_f(\varphi) = \int f\,\varphi\,dx$. Construct a sequence of function $\{\varphi_j\}$ in $\mathcal{D}$ that tends to zero in $\mathcal{S}$ but such that the sequence $\{L_f(\varphi_j)\}$ does not tend to zero. (Hint: Note that $L_f(\varphi_j)$ does tend to zero if $\{\varphi_j\}$ tends to zero in $\mathcal{D}$. So you need a sequence of functions $\{\varphi_j\}$ converging to zero in $\mathcal{S}$ but not in $\mathcal{D}$.)

5. Given a collection of numbers $\{c_\alpha\}$ where α runs over all multi-indeces of length less than or equal to N, show that there exists a function $\varphi \in \mathcal{D}(R^n)$ such that $\partial^\alpha\varphi(0) = c_\alpha$ for all α with $|\alpha| \leq N$. (Hint: You may need to use the product rule.)