MATH 890 – FOURIER ANALYSIS – F13

HW 3. More about the Fourier Transform

1. Compute the Fourier transform of the following functions

a) $f(x) = e^{-|x|}$, $x \in R$.
b) $f(x) = e^{-ax^2}$, $x \in R$.
c) $f(x) = e^{-ax}H(x)$, $x \in R$, where $H(x) = \chi_{(1,\infty)}(x)$ (the Heaviside function).
d) $f(x) = \chi_{[-A,A]}(x)$, $x \in R$, $A > 0$.
e) $f(x) = \chi_{[-A,A]^n}(x)$, $x \in R^n$, $A > 0$.
f) $f(x)$ continuous on R, constant on $[b, c]$, linear on $[a, b]$ and $[c, d]$ and zero outside $[a, d]$ for $-\infty < a < b < c < d < +\infty$.

2. Use Parseval's identity to compute

a) $\displaystyle\int_R \left(\frac{\sin x}{x}\right)^m dx$ for $m = 2, 3, 4$.

b) $\displaystyle\int_R \frac{1}{(1+x^2)^2}\, dx$

3. Assume that $f \in C(R)$ satisfies the size estimates

$$|f(x)| \leq C(1 + |x|)^{-2}$$

and

$$|\widehat{f}(\xi)| \leq C(1 + |\xi|)^{-2}.$$

Prove the Poisson summation formula:

$$\sum_{k\in Z} \widehat{f}(\xi - 2k\pi) = \sum_{k\in Z} f(k)e^{-ik\xi},$$

where both series converge uniformly on $(-\pi, \pi)$. In particular,

$$\sum_{k\in Z} \widehat{f}(2k\pi) = \sum_{k\in Z} f(k).$$

(Hint: use the size estimates to show that both series converge uniformly to continuous 2π-periodic functions. Then compute the Fourier coefficients of the left hand side.)

4. Compute the Fourier transform of $p.v.\ \frac{1}{x}$. (Hint: write $f_N(x) = \frac{1}{x}\chi_{\{1/n<|x|<N\}}(x)$, compute $< \widehat{f_N}, \varphi >$ for φ in $\mathcal{S}$. and pass to a limit. You may need to use $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$ as an improper Riemann integral.)

5. a) Show that f in $\mathcal{S}'(\mathbf{R}^n)$ is a polynomial if and only if supp $\widehat{f} = \{0\}$.

b) Let $\mathcal{S}_0(R^n)$ be the subspace of $\mathcal{S}(R^n)$ consisting of functions whose Fourier transform vanishes at zero to infinity order (i.e. $\partial^\alpha \widehat{f}(0) = 0$ for all α). Show that if f, g are in $\mathcal{S}'$, then $f = g$ when restricted to $\mathcal{S}_0$ if and only if $f = g + P$ where P is a polynomial. (Remark: Using a result from functional analysis called the Hahn-Banach Theorem, one can show that if we endow $\mathcal{S}_0$ with the topology of $\mathcal{S}$, then the dual of $\mathcal{S}_0$ can be identified with $\mathcal{S}'/\mathcal{P}$.)