MATH 890 – FOURIER ANALYSIS – F13

HW 5. More on Sobolev spaces

0. Recall our notation $P_s(\xi) = (1+|\xi|^2)^{s/2}$ and $\widehat{\Lambda^s f} = P_s \cdot \widehat{f}$.

1. Show that for any $s \in R$,

$$P_s(\xi) \leq P_s(\eta)\, 2^{|s|/2} P_{|s|}(\xi - \eta)$$

(Hint: By the triangular inequality $P_2(\xi) \leq 2\, P_2(\eta) P_2(\xi-\eta)$ and also $P_2(\eta) \leq 2\, P_2(\xi) P_2(\xi - \eta)$.)

2. Show that for $\varphi \in \mathcal{S}(R^n)$ the operator $T_\varphi f = \varphi f$ is bounded on every Sobolev space H^s. (Hint: First, note that T_φ is bounded on H^s iff $L = \Lambda^s \varphi \Lambda^{-s}$ is bounde on L^2. Next write $\widehat{Lg}$ as an integral operator of the form $\widehat{Lg}(\xi) = \int K(\xi,\eta)\widehat{g}(\eta)\, d\eta$. Using Problem 1 and the fact that φ is in $\mathcal{S}$, show that $\int |K(\xi,\eta)|\, d\eta \leq C$ and $\int |K(\xi,\eta)|\, d\xi \leq C$. Finally use Schur's test.)

3. Show that if $0 < \alpha < 1$ and $s = n/2 + \alpha$ then the map $x \to \delta_x$ is Lipschitz continuous of order α in the H^{-s}-norm, i.e.

$$\|\delta_x - \delta_y\|_{H^{-s}} \leq C_\alpha |x-y|^\alpha.$$

(Hint: Write $\|\delta_x - \delta_y\|^2_{H^{-s}}$ on the Fourier transform side and split the integral where $|\xi| \leq |x-y|^{-1}$ and $|\xi| \geq |x-y|^{-1}$. Use also that $|e^{ia\cdot\xi} - e^{ib\cdot\xi}| \leq C \min(2, |a-b|\,|\xi|)$.)

4. For $x \in R^n$, write $x = (x', x_n)$ with $x' = (x_1, \dots, x_{n-1})$ and identify R^{n-1} with the set points of the form $x = (x', 0)$. Show that the map $\gamma : \mathcal{S}(R^n) \to \mathcal{S}(R^{n-1})$ given by $\gamma u(x') = u(x', 0)$ satisfies

$$\|\gamma u\|_{H^{s-1/2}(R^{n-1})} \leq C \|u\|_{H^s(R^n)}$$

for all $s > 1/2$, and hence extends to a bounded linear map between such Sobolev spaces. (Hint: Show that $\widehat{\gamma u}(\xi') = c \int P_s(\xi)\widehat{u}(\xi', \xi_n)(1 + |\xi'|^2 + |\xi_n|^2)^{-s/2}\, d\xi_n$, where the $\widehat{\ }$ on γu is in R^{n-1} and the one on u is in R^n. Show then that

$$|\widehat{\gamma u}(\xi')|^2 \leq C(1+|\xi'|^2)^{1/2-s} \int P_s^2(\xi)|\widehat{u}(\xi', \xi_n)|^2\, d\xi_n.)$$