Summary of integration formulas

Math 123

1 Line integrals

Definitions

1. (Definition 5.2.1 in the book) The **line integral of a continuous scalar-valued function** $u : U \subset \mathbb{R}^n \to \mathbb{R}$ over a smooth path C lying in U and parametrized by $\mathbf{f}(t)$, $a \le t \le b$, is
$$\int_C u\,dL = \int_a^b u(\mathbf{f}(t))\|\mathbf{f}'(t)\|\,dt.$$
2. (Definition 5.2.2) The **line integral of a continuous vector field** $\mathbf{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ over a smooth path C lying in U and parametrized by $\mathbf{f}(t)$, $a \le t \le b$, is
$$\int_C \mathbf{F}\cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{f}(t))\cdot \mathbf{f}'(t)\,dt.$$

Remarks

1. Definition 1 is for **scalar-valued functions** and definition 2 is for **vector fields.**
2. If $u = 1$ we obtain the length of C
$$\text{Length}(C) = \int_C 1\,dL = \int_a^b \|\mathbf{f}'(t)\|\,dt.$$
3. If C represents a thin wire with density of mass δ then the total mass of the wire is
$$\text{Mass} = \int_C \delta\,dL = \int_a^b \delta(f(t))\|\mathbf{f}'(t)\|\,dt.$$
4. Another notation for $\int_C \mathbf{F}\cdot d\mathbf{x}$ is
$$\int_C F_1\,dx_1 + F_2\,dx_2 + \cdots + F_n\,dx_n,$$
where F_1, $F_2, \cdots F_n$ are the components of $\mathbf{F}$; that is $\mathbf{F} = (F_1, F_2, \cdots, F_n)$. In particular, in $\mathbb{R}^2$ and $\mathbb{R}^3$ we have, respectively,
$$\int_C F_1\,dx + F_2\,dy,$$
$$\int_C F_1\,dx + F_2\,dy + F_3\,dz.$$
(See page 286, formula 5.2.5 in the book).

2 Surface integrals.

Definitions

1. (Definition 5.5.2 in the book) Let M be a smooth surface in $\mathbb{R}^3$ that is parametrized by $\mathbf{f}(s,t)$, $(s,t) \in R \subset \mathbb{R}^2$. The **surface area of** M is

$$\sigma(M) = \iint_R \|\tfrac{\partial \mathbf{f}}{\partial s}(s,t) \times \tfrac{\partial \mathbf{f}}{\partial t}(s,t)\|\, ds\, dt.$$

2. (Definition 5.6.1 in the book) Let $g : U \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous **scalar-valued function** and let M be a smooth surface lying in U that is parametrized by $\mathbf{f}(s,t)$, $(s,t) \in R \subset \mathbb{R}^2$. The **surface integral of** g **over** M is

$$\iint_M g\, d\sigma = \iint_R g(\mathbf{f}(s,t)) \|\tfrac{\partial \mathbf{f}}{\partial s}(s,t) \times \tfrac{\partial \mathbf{f}}{\partial t}(s,t)\|\, ds\, dt$$

3. (Definition 5.6.2 in the book) Let $\mathbf{F} : U \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous **vector field** and let M be a smooth surface lying in U that is parametrized and oriented by $\mathbf{f}(s,t)$, $(s,t) \in R \subset \mathbb{R}^2$. The **surface integral of F over** M is

$$\iint_M \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \mathbf{F}(\mathbf{f}(s,t)) \cdot \left(\frac{\partial \mathbf{f}}{\partial s}(s,t) \times \frac{\partial \mathbf{f}}{\partial t}(s,t) \right) ds\, dt.$$

Remarks

1. Definition 2 is for **scalar-valued functions** in $\mathbb{R}^3$ and definition 3 is for **vector fields** in $\mathbb{R}^3$.

2. If, for example, the surface can be parametrized by x and y in some region $R \subset \mathbb{R}^2$ in the form $f(x,y) = (x, y, h(x,y))$, and $\mathbf{n}$ points up, then

$$\left(\frac{\partial \mathbf{f}}{\partial x}(x,y) \times \frac{\partial \mathbf{f}}{\partial y}(x,y) \right) = (-h_x, -h_y, 1),$$

and

$$\iint_M \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \mathbf{F}(x, y, h(x,y)) \cdot \Big(-h_x, -h_y, 1 \Big)\, dx\, dy.$$

3 Change of variables.

Formulas

1. (Theorem 5.7.1 in the book) Let g be a continuous function on a region $R \subset \mathbb{R}^2$ parametrized by $\mathbf{f}(s,t)$ for $(s,t) \in R^* \subset \mathbb{R}^2$, where $\mathbf{f}$ is a smooth one-to-one function. Then,
$$\iint_R g(x,y)\,dx\,dy = \iint_{R^*} g(\mathbf{f}(s,t)) \left| \frac{\partial \mathbf{f}(s,t)}{\partial(s,t)} \right| ds\,dt.$$

2. (Formula 5.8.1 in the book) Let g be a continuous function on a solid region $S \subset \mathbb{R}^3$ parametrized by $\mathbf{f}(s,t,u)$ for $(s,t,u) \in S^* \subset \mathbb{R}^3$, where $\mathbf{f}$ is a smooth one-to-one function. Then,
$$\iiint_S g(x,y,z)\,dx\,dy\,dz = \iiint_{S^*} g(\mathbf{f}(s,t,u)) \left| \frac{\partial \mathbf{f}(s,t,u)}{\partial(s,t,u)} \right| ds\,dt\,du.$$

Remarks

1. If we use polar coordinates in two dimensions then
$$\left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = r.$$

2. If we use cylindrical coordinates in three dimensions then
$$\left| \frac{\partial(x,y,z)}{\partial(r,\theta,w)} \right| = r.$$

3. If we use spherical coordinates in three dimensions then
$$\left| \frac{\partial(x,y,z)}{\partial(\rho,\theta,\Phi)} \right| = \rho^2 \sin\Phi.$$