Mathematics 143 – Spring 97

Group Project 1 **Due: Feb. 17, 1997**

Some applications of eigenvalues and eigenvectors

1. Introduction

An example of the use of eigenvalue problems in mathematics and the applied sciences is found in the study of system of linear differential equations. What follows is a very brief introduction.

Consider the *first-order differential equation*

$$y'(t) = ay(t) \quad \text{or} \quad y' = ay, \tag{1}$$

where a is a given constant, y is an unknown function of one variable and y' is its derivative. It is easy to verify that the function

$$y(t) = ce^{at} \tag{2}$$

for an arbitrary constant c is a solution of equation (1). Consider now a *system of first-order differential equations*

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\ y_2' &= a_{12}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{nn}y_n \end{aligned} \tag{3}$$

Here y_1, y_2, …, y_n, are n unknown functions of one variable. If A is the $n \times n$ matrix of coefficients in the above system and Y and Y' are the column vectors

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix}, \tag{4}$$

then the system (3) can be written as

$$Y' = AY. \tag{5}$$

Motivated by (2) we try to find a solution of the form

$$Y = \begin{pmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \\ \vdots \\ x_n e^{\lambda t} \end{pmatrix} = e^{\lambda t}X, \tag{6}$$

where $X = col(x_1, x_2, \ldots, x_n)$ and λ will have to be determined. If we compute the derivative of the vector function of t defined in (6), we obtain

$$Y' = \begin{pmatrix} \lambda e^{\lambda t} x_1 \\ \lambda e^{\lambda t} x_2 \\ \vdots \\ \lambda e^{\lambda t} x_n \end{pmatrix} = \lambda e^{\lambda t} X = \lambda Y. \tag{7}$$

Thus, if we choose λ to be an eigenvalue of the matrix A with associated eigenvector X, we finally get

$$AY = e^{\lambda t} AX = \lambda e^{\lambda t} X = \lambda Y = Y'. \tag{8}$$

This shows that,

if λ is an eigenvalue of the matrix A with associated eigenvector X,
then $Y = e^{\lambda t} X$ is a solution of the system
$Y' = AY$.

The analytic problem of solving the system of differential equations has been reduced to the algebraic problem of finding the eigenvalues and eigenvectors of a matrix.

2. Problems and questions

a) Show that systems like the one in (3) are *linear*. That is, if Y_I and Y_{II} are two solution of $Y' = AY$ then $c_I Y_I + c_{II} Y_{II}$ is also a solution for all constants c_I and c_{II}.

b) Use the method described in the introduction to find a solution to the system

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 3y_1 + 2y_2 \end{aligned} \tag{9}$$

c) Is the solution unique? Why?

d) Can you find a solution so that for $t = 0$, $y_1(0) = 6$ and $y_2(0) = 1$? Hint: use part a).

e) Try to make some conclusions about how many "different" (or, rather, linearly independent) solutions a system of n equations will have in general. How can you specify a particular one? (You don't need to give a rigorous proof of your conclusions; provide instead some arguments using your own words.)

3. An application to mixtures

Two tanks are connected as shown in the figure below. Initially, tank A contains 200 liters of water in which 60 grams of salt have been dissolved, and tank B contains 200 liter of pure water. Liquid is pump in and out of the tanks at the rates indicated in the diagram.

a) How much salt do you expect to find in the tanks after a very large time?

b) Determine the amount of salt in each tank at time t.

c) Is your answer to a) reflected in the solution to the problem you computed?

Some useful observations:

- The total amount of liquid in each tank will remain at 200 liters all the time since the amounts pumped in and out of the two connected tanks are the same.
- If $s_1(t)$ and $s_2(t)$ denotes the grams of salt in tanks A and B, respectively, at time t, then the rate at which salt is added to tank A is given by

  $$(5\mathrm{L/min}) \cdot \left(\frac{s_2(t)}{200} \mathrm{g/L} \right)$$

  and the rate at which salt is being pumped out of A is given by

  $$(20\mathrm{L/min}) \cdot \left(\frac{s_1(t)}{200} \mathrm{g/L} \right).$$

  Similarly for the tank B.
- The rate of change in the amount of salt at time t for each tank is equal to the rate at which salt is being added minus the rate at which is being pumped out.
- Try to combine the above information into some system of equations. Write down the system and solve it.

Hand-in all your answers and computations. Only one set of solutions per group.