MATH 647 – Spring 2001

Homework 8

Due Friday May 4.

a) Suppose that u and v are functions defined on a domain D in three dimensions and that their normal derivatives on the boundary ∂D satisfy

$$\frac{\partial u}{\partial n} = h = \frac{\partial v}{\partial n}$$

for a given function h. Let

$$E(v) = \frac{1}{2}\int\int\int_D |\nabla v|^2 \, dxdydz - \int\int_{\partial D} vh \, dS,$$

and similarly

$$E(u) = \frac{1}{2}\int\int\int_D |\nabla u|^2 \, dxdydz - \int\int_{\partial D} uh \, dS.$$

Show that

$$\begin{aligned} E(v) - E(u) = \frac{1}{2}\int\int\int_D |\nabla(v-u)|^2 \, dxdydz \\ + \int\int\int_D \nabla v \cdot \nabla u \, dxdydz - \int\int_{\partial D} v\frac{\partial u}{\partial n} \, dS. \\ + \int\int_{\partial D} u\frac{\partial u}{\partial n} \, dS - \int\int\int_D |\nabla u|^2 \, dxdydz. \end{aligned}$$

Hint: put $h = \frac{\partial u}{\partial n}$ in both $E(v)$ and $E(u)$ and use a by now familiar formula for $|\nabla(v-u)|^2$.

b) Show that if u solves the Neumann problem

$$\Delta u = 0 \text{ in } D$$
$$\frac{\partial u}{\partial n} = h \text{ on } \partial D$$

then $E(u) \leq E(v)$ for all v with $h = \frac{\partial v}{\partial n}$. Hint: Show that

$$E(v) - E(u) = \frac{1}{2}\int\int\int_D |\nabla(v-u)|^2 \, dxdydz \geq 0.$$

To do so, use that u is harmonic, part a), and the divergence theorem combined with the also familiar formula

$$\text{div}\,(f\nabla g) = f\Delta g + \nabla f \cdot \nabla g.$$