Mathematics 647 – Spring 2001

Special Project **Due: May 2, 2001**

Vibrations of a rectangular membrane

1. Goals

The purpose of this project is to further understand the solution of boundary value problems for the wave equation in in rectangular regions in two space dimensions. Some of the topics have been already discussed in class, but looking at some plots could help you better understand what is involved. The use of some computer software or graphing calculator will be needed for this project. To assist you in this regard, a Maple worksheet with some simple commands and suggestions has been placed in the course Web page. You can use such example or any other software or programs you may write.

2. Rules

You are expected to hand in written solutions which include answers to the questions and print outs of some plots. You may consult any book or publication you find of help. You are encouraged to discuss the problems with your classmates and you can do this project in a team of up to four people. If the project is done in collaboration, only one copy of the solutions needs to be handed in with all the names of the students participating in the team.

3. Problems

A. The Dirichlet problem for the wave equation.

Let S be the rectangle $(0,\pi)\times(0,\pi)$ and denote its boundary by ∂S. Consider the boundary value problem for a function $u(x,y,t)$,

$$u_{tt} = c^2(u_{xx}+u_{yy}) \quad \text{in} \quad S$$

$$u|_{\partial S} = 0 \quad \text{for all} \quad t$$

$$u(x,y,0) = f(x,y)$$

$$u_t(x,y,0) = g(x,y).$$

Solve the problem using Fourier series and separation of variables according to the following steps. SHOW YOUR WORK.

A1. Consider first solutions of the form $T(t)V(x,y)$ and arrive to a pair of equations, one involving T and another involving V. In particular, you should obtain

$$\Delta V = -\lambda V.$$

A2. Further separate variables in V to arrive to two identical eigenvalue problems. You may assume that the eigenvalues in each of these problems are positive. Denote them by

m^2 and n^2. Use the Dirichlet boundary condition for u to find the eigenfunctions and write the solutions $V_{m,n}(x,y) = X_m(x)Y_n(y)$ to the corresponding equations.

A3. Solve now the eigenvalue problem involving T obtained in A1.

A4. Write a series solution to the boundary value problem using the eigenfunctions obtained and the time dependent parts.

A5. Determine the coefficients in the series using the initial data. You may assume that you can manipulate the series solution term by term in any way you consider convenient in you calculations (i.e., integrating, differentiating, associating terms, etc...). Write explicit integral formulas for the coefficients involving the functions f and g.

B. The vibrating membrane

The solution $u(x,y,t)$ to the boundary value problem solved in Part A can be physically interpreted as the displacement or position at the point (x,y) and time t of a square membrane clamped on the four sides and set to vibrate by the initial displacement $f(x,y)$ and initial velocity $g(x,y)$. This is in analogy with the vibrating string interpretation of the same boundary value problem in one space dimension.

Consider for simplicity the case $f(x,y) = 0$. If you work out Part A successfully, you should arrive in this case to a series solution of the form

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{m,n}\sin(\sqrt{m^2+n^2}ct)\sin(mx)\sin(ny), \tag{1}$$

for appropriate coefficients $c_{m,n}$. Each of the functions $V_{m,n}(x,y) = \sin(mx)\sin(ny)$ is called a *mode* of vibration. The collection of points in S where each of these functions is zero is called the *nodal set* of $V_{m,n}$.

B1. Produce plots for the first few modes corresponding to $n = 1, 2, 3$ and $m = 1, 2$.

B2. Describe or produce a picture of the nodal set for the modes considered in B1.

B3. Consider the individual solutions of the wave equation with homogeneous boundary conditions

$$u_{m,n}(x,y,t) = \sin(\sqrt{m^2+n^2}ct)\sin(mx)\sin(ny).$$

What happens at the nodal points when this modes oscillate in time? Describe what happens with the adjacent cell determined by the nodal sets as the modes oscillate. (Hint: you may look at the plots of a couple of $u_{m,n}(x,y,t_0)$ functions for several values of fixed t_0 or produce an animations of them as t changes.)

B4. In analogy with the vibrating string, we can think of the vibrating membrane as the patch of a square drum and call the frequency $\sqrt{2}c$ of $\sin\sqrt{2}ct$ the *fundamental frequency* of the drum. The *overtones* are given by the frequencies $\sqrt{m^2+n^2}c$ associated to the other modes. How were the fundamental frequency and the frequencies of the overtones related in a vibrating string? Does the same relation holds in the drum? Write down the frequencies for a few first values of n and m.

This is suppose to explain why is easier to produce a musical note with a string instrument than with a drum. The relation (harmony) of the frequencies in the string is thought to be more pleasant to the ear. This may be the reason why you don't see many square drums.

C. A few more plots.

Assume the constant (wave speed) $c = 1$. Consider the approximation of the solution of the Dirichlet problem given by the partial sum using $n \leq N$ and $m \leq M$. Select some N and M based on the particular initial data given in each of the cases below. (Hint: in general use a small values, say $N = 5$ and $M = 5$ since there are NM coefficients to compute. Be alert though, that in some example you may need some larger M and/or N.) Compute using the formulas obtained in Part A or a computer the corresponding coefficients and produce plots of the resulting partial sum for values of $t = 0, \pi/4, \pi/2, 3\pi/4$.

C1. Initial Data:
$f(x, y) = (x - \pi)sin(x/2)siny$,
$g(x, y) = 0$.

C2. Initial Data:
$f(x, y) = (x - \pi)sin(x/2)sin(15y)$,
$g(x, y) = 0$.
What do you get if $N = 10$?

C3. Initial Data: (You hit the drum in the middle with a small square hammer.)
$f(x, y) = 0$,
$g(x, y) = -10(H(x - 0.45\pi) - H(x - 0.55\pi))(H(y - 0.45\pi) - H(y - 0.55\pi)$,
where H =Heaviside function in Maple. (Heaviside$(x - a)$ is the function that is equal to 1 for all $x \geq a$ and to 0 for all $x < a$. Therefore, $g(x, y)$ is a function that is equal to -10 on the square $[0.45\pi, 0.55\pi] \times [0.45\pi, 0.55\pi]$ and zero elsewhere.)

C4. In class we have solved the Neumann problem for the wave equation in S with initial data $f(x, y) = 0$ and $g(x, y) = sin^2 x$. Do not repeat the computations, but use the solution we obtained in class and produce plots of it for a few values of t. Give a physical interpretation of the solution thinking again in terms of a membrane. (Hint: plotting the solution for several values of t or animating it should give you the answer. If you use Maple you'll get an even more obvious picture if you use the option BOXED for the axes.