Bonus Homework

Math 766
Spring 2012

1) For $E_1, E_2 \subset \mathbb{R}^n$, define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

(a) Prove that if E_1 and E_2 are compact, then $E_1 + E_2$ is compact.
Proof: Since $E_1 + E_2 \subset \mathbb{R}^n$, it is sufficient to prove that $E_1 + E_2$ is closed and bounded.
$\underline{E_1 + E_2 \text{ is bounded:}}$ Since E_1 and E_2 are compact, they are bounded. So there exists $M > 0$ such that $|x| < M$ for all $x \in E_1$ and $|y| < M$ for all $y \in E_2$. Then for $z \in E_1 + E_2$, there exist $x \in E_1$ and $y \in E_2$ such that $z = x + y$. So

$$|z| \leq |x| + |y| < 2M.$$

Hence $E_1 + E_2$ is bounded.
$\underline{E_1 + E_2 \text{ is closed:}}$ Let $\{z_n\} \subset E_1 + E_2$ such that $z_n \to z$ for some $z \in \mathbb{R}^n$. There exist $x_n \in E_1$ and $y_n \in E_2$ such that $z_n = x_n + y_n$ for each $n \in \mathbb{N}$. Since E_1 is compact and $\{x_n\} \subset E_1$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x \in E_1$ such that $x_{n_k} \to x$ as $k \to \infty$. Now since E_2 is compact and $\{y_{n_k}\} \subset E_2$, there exists a subsequence $\{y_{n_{k_\ell}}\} \subset \{y_k\}$ and $y \in E_2$ such that $y_{n_{k_\ell}} \to y$ as $\ell \to \infty$. Then take $z_{n_{k_\ell}} \subset \{z_n\}$ and since $z_n \to z$

$$\begin{aligned} z = \lim_{\ell\to\infty} z_{n_{k_\ell}} &= \lim_{\ell\to\infty} x_{n_{k_\ell}} + y_{n_{k_\ell}} \\ &= x + y. \end{aligned}$$

But since $x \in E_1$ and $y \in E_2$, it follows that $z = x + y \in E_1 + E_2$. Therefore $E_1 + E_2$ is closed.
Hence $E_1 + E_2$ is closed and bounded. □

(b) There exists a closed set $E \subset \mathbb{R}$ such that $E + \mathbb{N}$ is not closed
Proof: Define

$$E = \{a_n\}_{n\in\mathbb{N}} \text{ where } a_n = \frac{1}{n} - n$$

For all $n \in \mathbb{N}$, $a_n > n_{n+1}$ and $a_n \to -\infty$ as $n \to \infty$. Then it follows that

$$E^c = (a_1, \infty) \cup \left[\bigcup_{n\in\mathbb{N}} (a_{n+1}, a_n)\right]$$

which is an open set. So E is closed. Now for each $n \in \mathbb{Z}$, we have $\frac{1}{n} \in E + \mathbb{N}$ since

$$\frac{1}{n} = \left(\frac{1}{n} - n\right) + n \in E + \mathbb{N}.$$

But $\frac{1}{n} \to 0$ as $n \to \infty$ and $0 \notin E + \mathbb{N}$. It can be seen that $0 \notin E + \mathbb{N}$ since for all $k, n \in \mathbb{N}$

$$k - n < \frac{1}{n} - n + k < k + 1 - n,$$

which implies that $E + \mathbb{N} \subset \mathbb{R} \backslash \mathbb{Z}$. Then $E + \mathbb{N}$ is not closed. □

2) If $\sum_{n=1}^{\infty} f_n$ converges pointwise to a continuous function f on $[0,1]$ and every f_n is continuous and non-negative on $[0,1]$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on $[0,1]$.
Proof: Let $\varepsilon > 0$ and define for $N \in \mathbb{N}$

$$K_N = K_N(\varepsilon) = \left\{ x : f(x) - \sum_{n=1}^{N} f(x) \geq \varepsilon \right\}.$$

For each $x \in [0,1]$, since $f_n \geq 0$ for all $n \in \mathbb{N}$

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \geq \sum_{n=1}^{N} f_n(x)$$

for every $N \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} f_n = f$ pointwise on $[0,1]$, for each $x \in [0,1]$ there exists $N_0 = N_0(x, \varepsilon) \in \mathbb{N}$ such that $N > N_0$ implies

$$f(x) - \sum_{n=1}^{N} f_n(x) = \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| < \varepsilon.$$

That is for each $x \in [0,1]$, there exists $N_0 = N_0(x, \varepsilon)$ such that $x \notin K_{N_0}$. Then

$$\bigcap_{N \in \mathbb{N}} K_N = \emptyset.$$

Also $K_{N+1} \subset K_N$ for all $N \in \mathbb{N}$ since for any $x \in K_{N+1}$ we have

$$f(x) - \sum_{n=1}^{N} f_n(x) \geq f(x) - \sum_{n=1}^{N+1} f_n(x) \geq \varepsilon.$$

Since f_n and f are continuous, so is $f - \sum_{n=1}^{N} f_n(x)$, and so

$$K_N = \left\{ x : f(x) - \sum_{n=1}^{N} f(x) \geq \varepsilon \right\}$$

is a closed set. Also $K_N \subset [0,1]$, so K_N is compact. Then K_N are a nested sequence of compact sets with empty intersection. Therefore by Cantor's intersection theorem, there exists $N_0 = N_0(\varepsilon) \in \mathbb{N}$ such that $K_{N_0} = \emptyset$. Then if $n > N_0$, then for all $x \in [0,1]$

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| = f(x) - \sum_{n=1}^{N} f_n(x) < \varepsilon.$$

Hence $\sum_{n=1}^{N} f_n(x)$ converges uniformly to f on $[0,1]$. □

3) Let $\psi \in C[0,1]$ and define for $f,g \in C[0,1]$

$$\rho_\psi(f,g) = \int_0^1 \psi(x)|f(x)-g(x)|dx.$$

- If $\psi(x) > 0$ for all $x \in [0,1]$, then ρ_ψ is a metric on $C[0,1]$
- If $\psi(x) = 0$ for $0 \le x \le 1/2$ and $\psi(x) = x - 1/2$ for $1/2 < x \le 1$, then ρ_ψ is not a metric on $C[0,1]$.

Proof: For any $f,g \in C[0,1]$, $\rho_\psi(f,g)$ is well defines since $\psi(x)|f(x)-g(x)|$ is a continuous and hence integrable function on $[0,1]$. Also

$$\rho_\psi(f,g) = \int_0^1 \psi(x)|f(x)-g(x)|dx = \int_0^1 \psi(x)|g(x)-f(x)|dx = \rho_\psi(g,f).$$

Now assume that $f \neq g$. Then there exists $x_0 \in [0,1]$ such that $f(x_0) \neq g(x_0)$. Since $|f(x)-g(x)|$ is continuous at x_0 and $|f(x_0)-g(x_0)| > 0$, there exists $\delta > 0$ such that for all $x \in [0,1]$ such that $|x-x_0| < \delta$

$$\big|\, |f(x)-g(x)| - |f(x_0)-g(x_0)| \,\big| < \frac{|f(x_0)-g(x_0)|}{2}$$

which implies that for $x \in [0,1]$ such that $|x-x_0| < \delta$

$$|f(x)-g(x)| \ge \frac{|f(x_0)-g(x_0)|}{2}$$

Also since ψ is continuous with $\psi(x) > 0$, there exists $x_1 \in [x_0-\delta, x_0+\delta] \cap [0,1]$ such that

$$\varepsilon = \inf_{x \in [x_0-\delta, x_0+\delta]} \psi(x).$$

Without loss of generality assume that $x_0 \neq 1$. Then

$$\begin{aligned}\rho_\psi(f,g) &= \int_0^1 \psi(x)|f(x)-g(x)|dx \\ &\ge \int_{x_0}^{x_0+\delta} \psi(x)|f(x)-g(x)|dx \\ &\ge \varepsilon\delta\frac{|f(x_0)-g(x_0)|}{2}.\end{aligned}$$

A symmetric argument holds if $x_0 = 1$ using that $x_0 \neq 0$. On the other hand if $f = g$, then $f(x) = g(x)$ for all $x \in [0,1]$ and

$$\rho_\psi(f,g) = \int_0^1 \psi(x)|f(x)-g(x)|dx = 0.$$

So $\rho_\psi(f,g) = 0$ if and only if $f = g$. Finally for $f,g,h \in C(0,1)$

$$\begin{aligned}\rho_\psi(f,g) &= \int_0^1 \psi(x)|f(x)-g(x)|dx \\ &\le \int_0^1 \psi(x)|f(x)-h(x)|dx + \int_0^1 \psi(x)|h(x)-g(x)|dx \\ &= \rho_\psi(f,h) + \rho_\psi(h,g).\end{aligned}$$

So ρ_ψ is a metric on $C(0,1)$. If $\psi(x)=0$ for $0\leq x\leq 1/2$ and $\psi(x)=x-1/2$ for $1/2<x\leq 1$, then to see that ρ_ψ is not a metric define

$$f(x)=\begin{cases} x-\frac{1}{2} & x\in[0,\frac{1}{2}] \\ 0 & x\in(\frac{1}{2},1] \end{cases} \qquad\qquad g(x)=0$$

which are both continuous on $[0,1]$. But the we have that $f\neq g$ and

$$\rho_\psi(f,g)=\int_{\frac{1}{2}}^{1}(x-\frac{1}{2})|f(x)-g(x)|dx=0.$$

So ρ_ψ cannot be a metric. □

4) Let (X,ρ) be a metric space and $E\subset X$ be closed. Then f defined

$$f(x)=\inf\{\rho(x,y):y\in E\}$$

is continuous and $f(x)=0$ if and only if $x\in E$.
Proof: Let $x\in X$ and $\varepsilon>0$. For any $z\in B_{\varepsilon/2}(x)$ and $y\in E$, by the triangle inequality we have

$$\rho(x,y)\leq\rho(x,z)+\rho(z,y)<\frac{\varepsilon}{2}+\rho(z,y)$$
$$\rho(y,z)\leq\rho(y,x)+\rho(x,z)<\rho(y,x)+\frac{\varepsilon}{2}.$$

Then

$$f(x)=\inf_{y\in E}\rho(x,y)\leq\rho(x,z)+\inf_{y\in E}\rho(z,y)\leq\frac{\varepsilon}{2}+f(z)<\varepsilon+f(z)$$
$$f(z)=\inf_{y\in E}\rho(y,z)\leq\inf_{y\in E}\rho(y,x)+\rho(x,z)\leq f(x)+\frac{\varepsilon}{2}<f(x)+\varepsilon.$$

Hence $z\in B_{\varepsilon/2}$ implies that $|f(x)-f(z)|<\varepsilon$ and f is continuous on X. If $f(x)=0$, then for all $n\in\mathbb{N}$ there exists $y_n\in E$ such that

$$\rho(x,y_n)<\inf_{y\in E}(x,y)+\frac{1}{n}=f(x)+\frac{1}{n}=\frac{1}{n}.$$

Then $\rho(x,y_n)\to 0$ as $n\to\infty$ which means that $y_n\to x$ in X as $n\to\infty$. But E is closed, so $x\in E$. On the other hand, if $x\in E$, then

$$0\leq\inf_{y\in E}\rho(x,y)\leq\rho(x,x)=0.$$

Then $f(x)=0$ if and only if $x\in E$. □

5) Let $f:U\to V$ be a continuously differentiable function between two nonempty open sets $U,V\subset\mathbb{R}^n$. Suppose that the Jacobian determinant of f is never zero on U, that $f^{-1}(K)$ is compact for any compact set $K\subset V$, and that V is connected. Then $f(U)=V$
Proof: First I claim that $f(U)$ is an open set in $\mathbb{R}^n$ and hence an open set in the subspace topology of V. Fix $y=f(x)\in f(U)$ where $x\in U$. Since $\nabla f(x)\neq 0$, by the inverse function theorem, there exist open neighborhoods $U_x\subset U$ of x and $V_y\subset V$ of y such that f is 1-1 from U_x onto V_y and f^{-1}

is continuously differentiable on V_y. That is $V_y = f(U_x)$ is an open neighborhood of y contained in $f(U)$. Hence $f(U)$ is open. Now I claim that $f(U)$ is a closed set in the subspace topology of V. Let $y_n \in f(U)$ where $y_n \to y$ as $n \to \infty$ for some $y \in V$. Define $K = \{y_n\}_{n\in\mathbb{N}} \cup \{y\} \subset V$. Since $y_n \to y$, it follows that K is compact. There exist $x_n \in f^{-1}(K) \subset U$ such that $f(x_n) = y_n$. By assumption $f^{-1}(K) \subset U$ is also compact, so there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ converging to some $x' \in f^{-1}(K) \subset U$ as $k \to \infty$. Since f is continuous,

$$f(x') = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} y_n = y.$$

Therefore $y \in f(U)$, which proves that $f(U)$ is closed in V. Then $f(U)$ and $V\backslash f(U)$ are open sets in V. Moreover $f(U) \cap (V\backslash f(U)) = \emptyset$ and $f(U) \cup (V\backslash f(U)) = V$. Since V is connected, either $f(U) = \emptyset$ or $V\backslash f(U) = \emptyset$. Since U is nonempty, $f(U) \neq \emptyset$. Therefore $V\backslash f(U) = \emptyset$ and it follows that $V \subset f(U)$. The oposite inclusion, $f(U) \subset V$, follows trivially since $f : U \to V$. So we have that $f(U) = V$. □