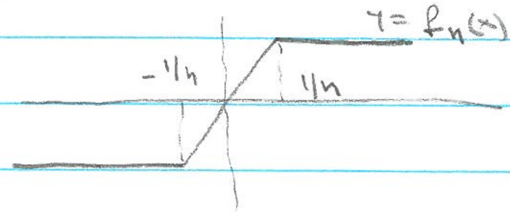
PRACTICE Pb 5

$$X = C[-1,1] = \{ f : [-1,1] \to \mathbb{R} \text{ s.t. } f \text{ is cont.} \}$$

$$\rho(f,g) = \| f - g \|_1 = \int_{-1}^{1} |f(x) - g(x)| \, dx$$

$\{f_n\}$ THE SEQUENCE OF Pb 4



a) SHOW THAT $\{f_n\}$ IS CAUCHY IN THE NEW METRIC ρ:

LET $m > n$, SO $\frac{1}{m} < \frac{1}{n}$

$$\rho(f_m, f_n) = \int_{-1}^{1} |f_m(x) - f_n(x)| \, dx$$

$$\leq \int_{-1/n}^{0} (f_n(x) - f_m(x)) \, dx + \int_{0}^{1/n} (f_m(x) - f_n(x)) \, dx$$

$$\leq \int_{-1/n}^{0} (0 - (-1)) \, dx + \int_{0}^{1/n} (1 - 0) \, dx$$

$$= \frac{2}{n}$$

SO GIVEN $\varepsilon > 0$ IF $m > n > n_0 \geq \frac{2}{\varepsilon}$

$\Rightarrow \rho(f_m, f_n) < \varepsilon$

$\Rightarrow \{f_n\}$ IS A CAUCHY SEQUENCE.

b) SHOW THAT $\{f_n\}$ CANNOT CONVERGE TO A FUNCTION IN X WITH THIS METRIC:

SUPPOSE $f_n \to f$ IN (X, ρ), SO IN PARTICULAR f IS CONT.

LET $x_0 \in [-1, 1]$, $x_0 > 0$.

IF $f(x_0) > 1$, THEN $f(x_0) = 1 + \varepsilon$ FOR SOME $\varepsilon > 0$. BUT SINCE f IS CONT.

$\exists\, \delta > 0$ S.t. $0 < x_0 - \delta < x_0 < x_0 + \delta < 1$

AND $f(x) > 1 + \varepsilon/2 \quad \forall\, x \in (x_0 - \delta, x_0 + \delta)$.

ALSO, $\exists\, n_0 \in \mathbb{N}$ S.t. $\frac{1}{n} < x_0 - \delta \quad \forall\, n \geq n_0$

$\Rightarrow f_n(x) = 1 \quad \forall\, x \in (x_0 - \delta, x_0 + \delta), \forall\, n \geq n_0$

THEN, $\rho(f_n, f) = \int_{-1}^{1} |f_n(x) - f(x)|\, dx$

$$\geq \int_{x_0 - \delta}^{x_0 + \delta} |f_n(x) - f(x)|\, dx \geq \int_{x_0 - \delta}^{x_0 + \delta} \varepsilon/2 \, dx$$

$$= \underbrace{\varepsilon \cdot \delta}_{\text{FIXED}} > 0 \quad \forall\, n \geq n_0$$

THIS CONTRADICTS THE FACT THAT $f_n \to f$ SO $f(x_0) \leq 1$. SIMILARLY WE CAN SHOW THAT $f(x_0)$ CANNOT BE SMALLER THAT 1. AND SO $f(x)$ MUST BE 1 $\forall\, x > 0$.

ALSO IN SIMILAR WAY $f(x) = -1 \quad \forall\, x < 0$

BUT THERE IS NO CONT. FUNCTION S.t.

$$f(x) = \begin{cases} -1 & \forall\, x < 0 \\ 1 & \forall\, x > 0 \end{cases}$$

. SO WE HAVE A CONTRAD. AND f_n CANNOT CONV. IN X.