MATH 766 – Spring 2012

Additional practice problems for Exam 1

- The exam will be based on the material in the book form Sections 7.1, 7.2, 7.3, 8.1, 8.2, 8.3, and 10.1, that we covered in class. (Note that some of the results in 9.1 are particular case of the results for metric spaces, so you may want to read them too). If in doubt please ask me.
- Of course, you need to know all the definitions of the concepts introduced and be able to state them.
- Study the proofs of the results presented in the lectures. Becoming acquainted with the proofs can help you prove similar related results.
- Prove and/or complete the properties and simple facts left as exercises in class.
- Review your homework and make sure you know how to do the all the problems assigned.
- Do the following practice problems. Warning: These are just practice problems to help you further understand some of the concepts we covered so far. This is not a sample test.

Additional problems from the book
Pbs. 7.1.6, 7.2.4 ; 7.3.9; 8.1.9; 8.3.1, 8.3.2; 10.1.1, 10.1.2, 10.1.3, 10.1.5.

Further problems
Problem 1.
A function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a norm if

- $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$ and $\|x\| = 0$ iff $x = 0$.
- $\|\alpha x\| = |\alpha| \ \|x\|$ for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$.

Show that if $\|\cdot\|$ is a norm, then $\|x - y\| \geq | \, \|x\| - \|y\| \, |$ for all $x, y \in \mathbb{R}^n$.

Problem 2.
Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1, 2x_2, 3x_3)$. Compute the operator norm $\|T\|$.

Problem 3.
Let X be a metric space. True or False? (If True just say so; if False give a counterexample.)

a) Every bounded sequence in X has a convergent subsequence.

b) Every Cauchy sequence in X is bounded.

c) Every Cauchy sequence in X has a convergent subsequence.

Problem 4.
Let $X = C[a,b]$ be the space of real valued continuous functions on the interval $[a,b]$ endowed with the usual metric $\rho(f,g) = \|f-g\|$, where $\|f\| = \sup_{x\in[a,b]} |f(x)|$.

a) Let $\{f_n\}_n$ be a sequence converging in X to the function $f(x) \equiv c$, for some constant $c > 0$. Show that there exist $N > 0$ such that $\inf_{n\geq N} \inf_{x\in[a,b]} f_n(x) > 0$.

b) Consider the sequence of elements $\{f_n\}$ in $X = C[-1,1]$, where each f_n is the continuous and odd function that is linear on $[-1/n, 1/n]$, and equal to 1 on $[1/n, 1]$. Draw a picture. Is this sequence a Cauchy sequence in X? Justify your answer.

Problem 5.
Let $X = C[-1,1]$ be the space of real valued continuous functions on the interval $[-1,1]$, with the new metric $\rho_1(f,g) = \|f-g\|_1$, where now $\|f\|_1 = \int_{-1}^{1} |f(x)|\, dx$. (By Problem 1 this is a metric space).

a) Consider the sequence of functions $\{f_n\}$ of Problem 5 b). Show that the sequence is Cauchy in the new metric.

b) Show that the sequence cannot converge to a continuous function in the new metric.