MATH 810 – REAL ANALYSIS

HW 3. Measurable functions

Due 10/14/15

NOTE: As explained in class, given measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$, the book by Folland defines a function $f : X \to Y$ to be $(\mathcal{M}, \mathcal{N})$ measurable if $f^{-1}(A) \in \mathcal{M}$ for all $A \in \mathcal{N}$. In the particular case of $Y = \mathbf{R}^*$, we have defined measurability to mean that the subsets of X given by, say, $f_{<\alpha}$ are measurable for all $\alpha \in \mathbf{R}$. *This is the definition you need to use for this HW.* We will show in particular (see problem 2) that our notion of measurability is equivalent with being $(\mathcal{M}, \mathcal{B})$ measurable in Folland's sense, where $\mathcal{B}$ is the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to R^*$ be a measurable function. Show that the collection $A = \{E \subset R^* : f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to R^*$ be a measurable function. Show $f^{-1}(B)$ is a measurable set for every Borel set B.

3. Let $f : R \to R$ be a continuous function. Show that f is measurable.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, let $f : X \to R$ be a measurable function, and let $g : R \to R$ be a continuous function. Show that $g \circ f$ is measurable.

5. Recall the construction of the Cantor set C given in class and of the Cantor function $f : [0, 1] \to [0, 1]$ which is non-decreasing, continuous and onto.

a) Define $g : [0, 1] \to [0, 1]$ by $g(x) = (f(x) + x)/2$. Show that g is strictly increasing, continuous and onto. (Therefore it has a continuous inverse g^{-1}.)

b) Show that $g(C)$, where C is the Cantor set, is a measurable set of positive measure. (Hint: Estimate the the measure of $g([0, 1] - C)$.)

c) You may assume that every set of positive measure has a non-measurable subset (actually, this follows from the exercises in Homework 1). Using this show that there exist a measurable set $M \subset [0, 1]$ such that $g(M)$ is not measurable. (Hint: g^{-1} is continuous and C has measure zero.)

d) Show that there exist a measurable set M which is not a Borel set.

e) Show that the composition of measurable function on $\mathbf{R}$ with Lebesgue measure (and with our definition of measurability) is not always measurable. (Hint: Consider $\chi_M \circ g^{-1}$, where M is the set in d).)