MATH 810 – REAL ANALYSIS

HW 5. More about integration and modes of convergence

Due 11/11/15

1. Let $f : [0, a] \to \mathbf{R}$ be a measurable function. Show that if f is Lebesgue integrable on $[0, a]$, then

$$\lim_{n\to\infty} \int_{(1/n,a]} f(x)\, dm(x) = \int_{[0,a]} f(x)\, dm(x).$$

2. Let $f : [0, a] \to \mathbf{R}$ be a measurable function. Show that if f is Riemann integrable on $[b, a]$ for all $b > 0$ and

$$\int_{0+}^{a} |f(x)|\, dx = \lim_{b\to 0+} \int_{b}^{a} |f(x)|\, dx < \infty$$

exists as an improper Riemann integral, then f is Lebesgue integrable on $[0, a]$ and

$$\int_{[0,a]} f(x)\, dm(x) = \int_{0+}^{a} f(x)\, dx.$$

3. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 1/\sqrt{|x|}$ for $|x|$ in $(0, 1)$ and $f(x) = 1/x^2$ for $|x|$ in $(1, +\infty)$. Compute the Lebesgue integral of f on $\mathbf{R}$. Justify your computations. (Remark: It does not matter how f is defined at 0 or 1.)

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences of measurable functions. Show that if $f_n \to f$ and $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure. How about $f_n \cdot g_n$? Justify your answer.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. For f and g, measurable functions on X, defined

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|}\, d\mu.$$

Show that if we identify functions that are equal a.e. then ρ is a metric on the space of measurable function and $f_n \to f$ in this metric if and only if $f_n \to f$ in measure.
(It is time to start reviewing about metric spaces if you have forgotten about them.)