MATH 810 – REAL ANALYSIS

Final Assingment

L^p spaces

Due by 10:00 a.m. on 12/15/15

(You may assume that all the measure spaces in the questions below are σ-finite)

1. Let (X, M, μ) be a measure space. Let $1 < p_i < \infty$, $i = 1, \cdots, n$ be real numbers satisfying

$$\sum_{i=1}^{n} \frac{1}{p_i} = 1.$$

Show that if f_i is in $L^{p_i}(X)$, $i = 1, \cdots, n$, then $\prod_i f_i$ is in $L^1(X)$ and

$$\| \prod_{i=1}^{n} f_i \|_{L^1(X)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(X)}.$$

Hint: Döes the case n = 2 look familiar? (ö is not a misprint). Then use induction.

2. Let $1 < p, q < \infty$, $p \neq q$, and let $L^p(R)$ and $L^q(R)$ be defined with respect to the Lebesgue measure. Show that there exists a function f in $L^p(R)$ such that f is not in $L^q(R)$.

3. Let $(X, \mathcal{M}, \mu)$ be a $\sigma-$finite measure space. Write $X = \cup_{j=1}^{\infty} E_j$ where $0 < \mu(E_j) < \infty$ and $E_j \subset E_{j+1}$ for all j. Let $f \in L^p(X)$ for some $1 \leq p < \infty$, and define $f_j = f\chi_{E_j}$. Show that $f_j \to f$ in L^p. Show also that, in general, the result is not true if $p = \infty$.

4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. For $1 < p, q < \infty$, and $f : X \times Y \to \mathbf{C}$ $\mu \times \nu$ measurable, define

$$\|f\|_{L^pL^q} = \left(\int_Y \left(\int_X |f(x, y)|^q \, d\mu(x) \right)^{p/q} d\nu(y) \right)^{1/p}$$

and consider the space of functions (equivalent classes of functions as always) L^pL^q defined by the above norm (you can take for granted that it is a norm). Show that if $f \in L^pL^q$ and $g \in L^rL^s$ with $1 = 1/p + 1/r$ and $1 = 1/q + 1/s$, then

$$\int_{X \times Y} |f(x, y)g(x, y)| \, d\mu \times d\nu < \infty.$$

5. Let (X, M, μ) be a measure space with $\mu(X) < \infty$. Show that $\|f\|_{L^\infty(X)} = \lim_{p\to\infty} \|f\|_{L^p(X)}$. Hint: For $\epsilon > 0$, small consider the set $A_\epsilon = \{x : |f(x)| > \|f\|_{L^\infty} - \epsilon\}$. Then A_ϵ has positive measure. Show that $\|f\|_{L^p} \geq (\|f\|_{L^\infty} - \epsilon)(\mu(A_\epsilon))^{1/p}$. Conclude that $\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$. A converse inequality $\limsup_{p\to\infty} \|f\|_{L^p} \leq \|f\|_{L^\infty}$ is easy.