MATH 890 – FOURIER ANALYSIS – F13

HW 2. More about distributions

1. a) Let $\{f_j\}$ be a sequence of functions in $L^1_{loc}(\Omega)$ such that $f_j \to f$ a.e. in Ω. Assume that for every compact $K \subset \Omega$ there exists a function $g_k \in L^1(\Omega)$ such that $|f_j(x)| \leq g_K(x)$ a.e $x \in K$ for all j. Prove that $f_j \to f$ in $\mathcal{D}'(\Omega)$.

b) Construct a sequence of L^1_{loc} functions in R converging to 0 a.e., but whose limit in $\mathcal{D}'(R)$ is not 0. (Hint: keep in mind part a).)

c) Construct a sequence of $L^2(R)$ function that converges to zero in $\mathcal{D}'(R)$ but not in $L^2(R)$.

2. Let $\varphi \in \mathcal{D}(R)$, $\varphi(x) \geq 0$, $\operatorname{supp}\varphi \subset [-1,1]$, $\int \varphi dx = 1$. Show that $\varphi_\epsilon(x) = 1/\epsilon\varphi(x/\epsilon)$ converges to δ in $\mathcal{D}'(R)$ as ϵ goes to zero. (Hint: check that for each $\psi \in \mathcal{D}(R)$, $\langle \delta - \varphi_\epsilon, \psi\rangle$ tends to zero.)

3. a) Prove that a function $\psi \in \mathcal{D}(R)$ is the derivative of another function in $\mathcal{D}(R)$ if and only if $\int \psi(x)dx = 0$.

b) Prove that if $u \in \mathcal{D}'(R)$ and $u' = 0$ then u is constant, in the sense that $\langle u, \psi\rangle = c\int \psi dx$ for some constant c. (Hint: use part a). Note that if $\Phi \in \mathcal{D}$ and $\int \Phi dx = 1$, then $\varphi(x) - \Phi(x)\int \varphi(t)dt$ has mean zero.).

4. Let f be a continuous function in $R - \{x_0\}$ whose derivative f' in the classical sense is also a continuous function in $R - \{x_0\}$. Assume further that the lateral limits $f(x_0\pm)$ and $f'(x_0\pm)$ exists and are finite. Since f is in particular locally integrable, $f \in \mathcal{D}'(R)$. Let Df be the distributional derivative of f in $\mathcal{D}'(R)$. Show that $Df = f' + (f(x_0+) - f(x_0-))\delta_{x_0}$.

5. Let f be a $L^1_{loc}(R^n)$. It is easy to check that the distribution $L_{f,\alpha} \in \mathcal{D}'(R^n)$ defined by

$$L_{f,\alpha}(\varphi) = \int f(x)\partial^\alpha\varphi(x)\,dx$$

has order at most $N = |\alpha|$

a) Show, however, that if $f \in C^N$ then $L_{f,\alpha}$ has order zero.

b) Show that if $f(x) = |x|^{-1/2}$, then $L_{f,1} \in \mathcal{D}'(R)$ has order one. (Hint: consider –for example using appropriate "dilations"– a sequence of functions $\{\varphi_j\}$ such that $\{\|\varphi_j\|_{L^\infty}\}$ remains bounded but $\{|L_{f,1}(\varphi_j)|\}$ tends to infinity.)

6. (Optional) The function $f(x) = |x|^{-1}$ is in $L^1_{loc}(R^3)$ and so it can be viewed as a distribution in $\mathcal{D}'(R^3)$. Show that $\Delta f = -4\pi\delta$ in the sense of distributions, where $\Delta = \sum_{j=1}^3 \partial_j^2$. (Hint: show first that $\Delta f(x) = 0$ in the classical sense for $x \neq 0$. Next, for $\varphi \in \mathcal{D}(R^3)$ write $\int f\,\varphi\,dx$ as $\lim_{\epsilon\to 0}\int_{\epsilon<|x|<A}$. Finally use Green's identity

$$\int_\Omega (v\Delta u - u\Delta v)\,dx = \int_{\partial\Omega}(v\partial_n u - u\partial_n v)\,ds,$$

where u and v are smooth functions, Ω is a smooth domain with boundary $\partial\Omega$, ∂_n is the normal derivative to the surface $\partial\Omega$, and ds is surface measure.)

(HW continues in next page)

7. (Just for reading) See: De Pauw, Thierry and Pfeffer, Washek F, *Distributions for which* $\operatorname{div} v = F$ *has a continuous solution,* Comm. Pure Appl. Math. 61 (2008), no. 2, 230-260.

Let F be a distribution in R^n so that, in the sense of distributions, $F = \operatorname{div} v$ for some continuos vector field v. Show that for each $\epsilon > 0$ there exists a constant C_ϵ such that

$$|\langle F, \varphi \rangle| \leq C_\epsilon \|\varphi\|_{L^1} + \epsilon \|\nabla \varphi\|_{L^1}$$

for all $\varphi \in \mathcal{D}(R^n)$ with $\operatorname{supp} \varphi \subset B(0, 1/\epsilon)$.

Remark: In the above mentioned article it is showed that this necessary condition is also sufficient to have a continuous solution v of the equation $F = \operatorname{div} v$. But that is much harder to prove.